Lagrangian Lecture III

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A problem with single particles

Single particle dispersion has *generic limits*:

$$lim_{t \to 0} \; K = \nu^2 t$$

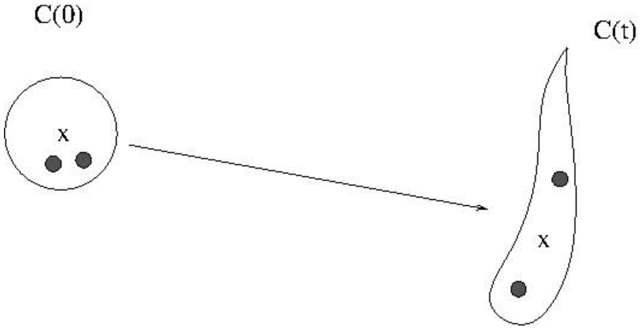
and

$$lim_{t \to \infty} \; K = \nu^2 T_L$$

→ True for a variety of flows

Not useful for distinguishing flow characteristics (large scale turbulence, small scale, time dependent flow, etc.)

Tracer spread



Tracer spread

First moment reflects the motion of the center of mass:

$$\frac{d}{dt} < x > = < U >$$

Previously, found the mean of x^2:

$$\frac{d}{dt} < x^2 > = < xU > + 2K_x$$

For a constant mean flow, $U = a$:

$$< x^2 > = a^2 t^2 + 2K_x t$$

Tracer spread

But the proper second moment (the variance) is:

$$D =< (x- < x >)^2 >=< x^2 > - < x >^2$$

For the constant mean flow, this is:

$$D = a^2 t^2 + 2K_x t - a^2 t^2 = 2K_x t$$

So the variance only increases due to diffusion.

Tracer spread

However, if $U = a + by$, we have:

$$
\begin{aligned}
D &= a^2 t^2 + \frac{2}{3} b^2 K_y t^3 + 2 K_x t - a^2 t^2 \\
&= \frac{2}{3} b^2 K_y t^3 + 2 K_x t
\end{aligned}
$$

$\rightarrow$ The shear increases the variance

Tracer spread

Can show that for a cloud:

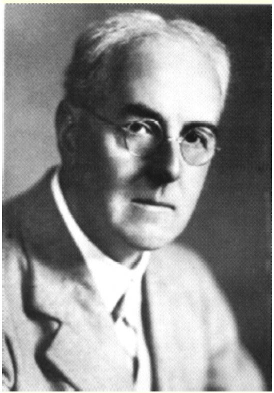
$$< (\vec{r} - < \vec{r} >)^2 > = < (r_i - r_j)^2 >$$

• The variance is equal to the mean square separation of all *pairs* of particles in the cloud

Can measure tracer spreading with pairs of drifters

LaCasce (2008)

Richardson (1926): Atmospheric diffusion shown on a distance-neighbour graph



Richardson (1926)

$$\frac{\partial}{\partial t} C = \frac{\partial}{\partial y}(\kappa \frac{\partial}{\partial y} C)$$

Richardson, 1926

Diffusivity

Richardson (1926):

$$\kappa \propto y^{4/3}$$

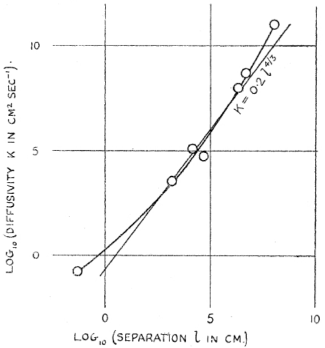
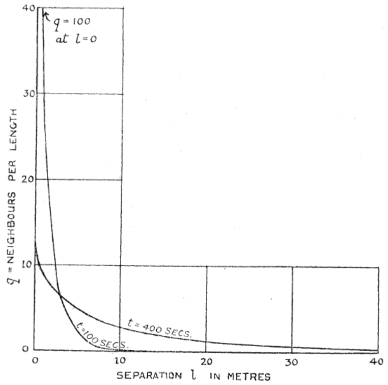


Fig. 8.

Fokker-Plank equation

$$\frac{\partial}{\partial t}C = \frac{\partial}{\partial y}(\beta y^{4/3}\frac{\partial C}{\partial y})$$



Two-particle statistics

Relative dispersion:

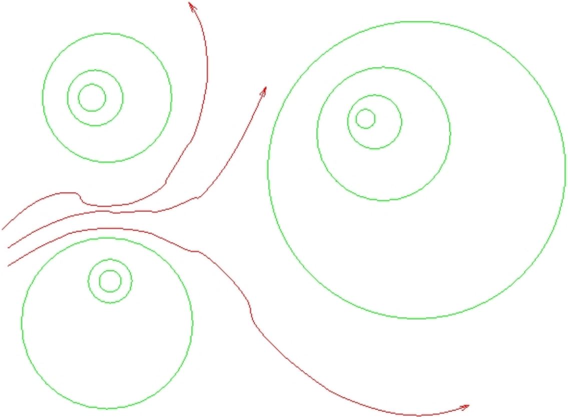
$$< \delta r^2 > \equiv \frac{1}{N_{pairs}} \sum_{i \neq j} |\vec{r}_i - \vec{r}_j|^2$$

Relative diffusivity:

$$\begin{aligned} K_2 &= \frac{1}{2} \frac{d}{dt} < \delta r^2 > \\ &= < \delta v(t) \, \delta r(t) > \\ &= \int_0^t < \delta v(t) \delta v(t') > \; dt' - < \delta v(t) \delta r(0) > \\ &\approx \int_0^t < \delta v(t) \delta v(t') > \; dt' \end{aligned}$$

Uncorrelated motion

At large separations, the pair velocities are *uncorrelated*



Uncorrelated motion

Assuming a homogeneous flow:

$$< \delta v(t)\delta v(t') > =< (v_i(t) - v_j(t))(v_i(t') - v_j(t')) >$$
$$\approx< v_i(t)v_i(t') > + < v_j(t)v_j(t') >$$
$$= 2\nu^2$$

This implies:

$$lim_{t\rightarrow\infty}\ K_2 = 2K_1$$

$\rightarrow$ The single particle diffusivity is recovered at long times

Relative diffusivity

However we can't really conclude:

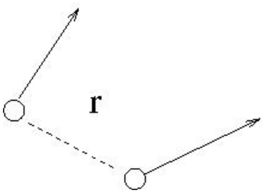
$$lim_{t \to 0} \;\; < \delta v(t) \delta v(t') >= Const.$$

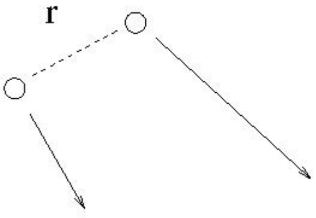
$\rightarrow$ The initial behavior reflects *statistics of the flow itself*

Structure functions

For a homogeneous flow:

$$\delta v_L(r) = \delta v_E(r)$$





Structure functions

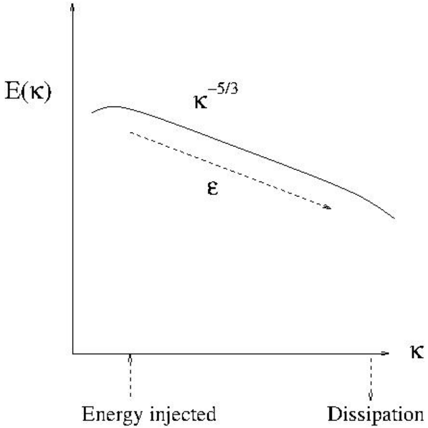
This allows us to exploit turbulence theory to predict the dispersion

In fact Kolmogorov's (1941) theory was formulated in terms of structure functions:

$$S_n(r) \equiv < |\vec{u}(x+r) - \vec{u}(x)|^n >$$

In the inertial range of 3-D turbulence, the energy dissipation rate, ϵ, is assumed to be constant across scales

Structure functions



Structure functions

The dissipation rate has units:

$$\epsilon \propto \frac{\partial E}{\partial t} \propto \frac{m^2}{sec^3}$$

If this is the only important parameter:

$$S_2 = < |\vec{u}(x+r) - \vec{u}(x)|^2 > \propto \epsilon^{2/3} r^{2/3}$$

This is Kolmogorov's "2/3 Law"

3-D turbulence

For homogeneous flows, the Lagrangian velocity difference behaves the same:

$$< \delta v_l(r)^2 > \propto \epsilon^{2/3} r^{2/3}$$

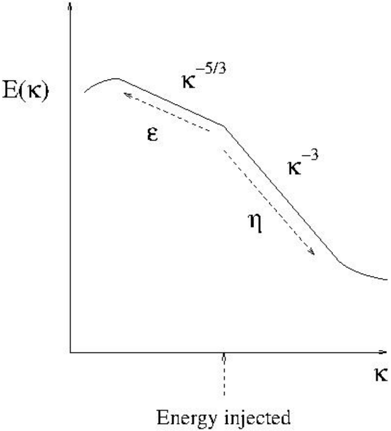
Similarly, the diffusivity is:

$$K_2 \propto \frac{m^2}{sec} \propto \epsilon^{1/3} r^{4/3}$$

$\rightarrow$ Consistent with Richardson's (1926) observations

Obukhov (1941), Batchelor (1952)

2-D turbulence



2-D turbulence

In the energy cascade:

$$K_2 \propto \epsilon^{1/3} r^{4/3}$$

In the *enstrophy* cascade:

$$\eta \propto \frac{\partial Z}{\partial t} \propto \frac{1}{sec^3}$$

So:

$$< \delta v_l(r)^2 > \propto \eta^{2/3} r^2$$

and:

$$K_2 \propto \frac{m^2}{sec} \propto \eta^{1/3} r^2$$

Relative dispersion

These have very different dispersion. If:

$$K_2 = \frac{1}{2}\frac{d}{dt} < r^2 > \propto \epsilon^{1/3} r^{4/3}$$

then can show:

$$< r^2 > \propto \epsilon t^3$$

But if:

$$K_2 = \frac{1}{2}\frac{d}{dt} < r^2 > \propto \eta^{1/3} r^2$$

then find:

$$< r^2 > \propto exp(8\eta^{1/3} t)$$

General relation

It's possible to derive the diffusivity given any power law spectrum for KE. If $E(k) \propto k^{-n}$, then:

$$K_2 \propto r^{(n+1)/2}$$

However, if the spectrum is *steeper* than k^{-3}, then:

$$K_2 \propto r^2$$

and the relative dispersion increases exponentially in time

Bennett (1984)

Some implications

• Relative dispersion applies to error growth—if the particle is moved a small distance from its initial position, how quickly does the error grow?

• Exponential growth implies a *sensitive dependence on initial conditions*. This is Lagrangian chaos

• Relative dispersion is exponential at sub-grid scales in models

Loch Long, Scotland

In the sea we used floats of parsnip because it is easily visible, and because it is almost completely immersed so as not to catch the wind which, moreover, was slight. The floats were about 2 cm in diameter. An optical device was used for measuring the distance l in a fixed azimuth. The observations were made in latitude 56°0′N, longitude 4°54′W from Blairmore Pier, Loch Long, Scotland, on 6 January 1948, where the sea water was about two meters deep. In order to eliminate any change in $F(l)$ with time, we observed alternately with large and small l_0, as may be seen from table 1. From equation (6) the function $F(l)$ was computed separately for the wide and close pairs:

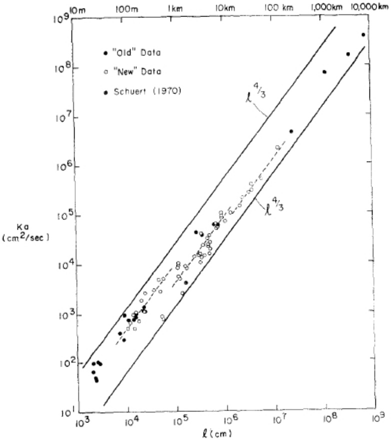
	wide pairs	close pairs	unit
l	187.7	26.7	cm
$F(l)$	84.3	6.4	cm² sec⁻¹

The power law which fits these data is

$$F(l) = 0.07\, l^{1.4}. \tag{9}$$

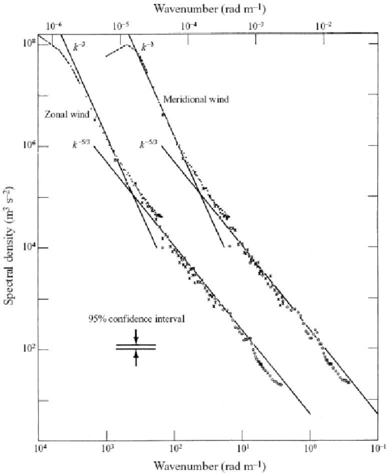
Richardson and Stommel, 1948

Surface dispersion



Okubo, 1970

Atmospheric spectra



EOLE

483 balloons in the Southern Hemisphere stratosphere

EOLE balloons

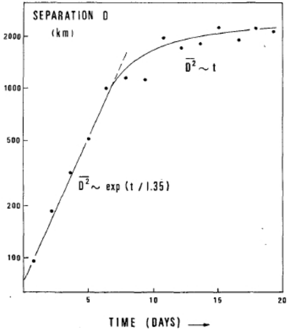


Fig. 8. Root mean square separation of the original pairs of balloons released during the EOLE experiment, as a function of time after launch.

Morel and Larcheveque, 1974

TWERLE balloons

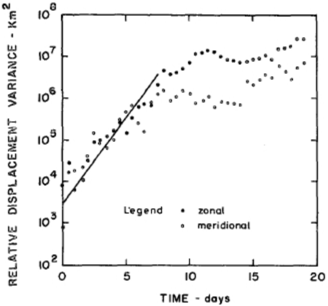
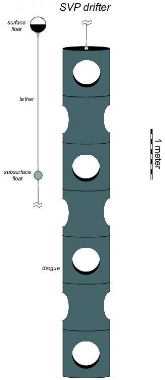


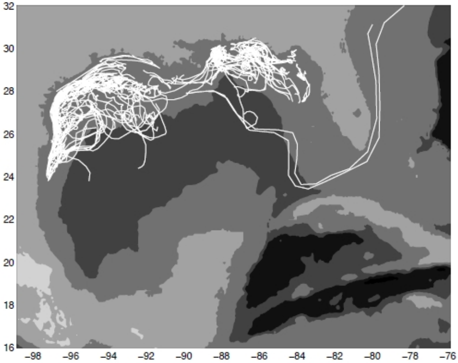
FIG. 6. The mean-square relative displacement components for midlatitudes releases on a log-linear scale. The straight line indicates an exponential region.

Er-el and Peskin, 1981

Surface drifters

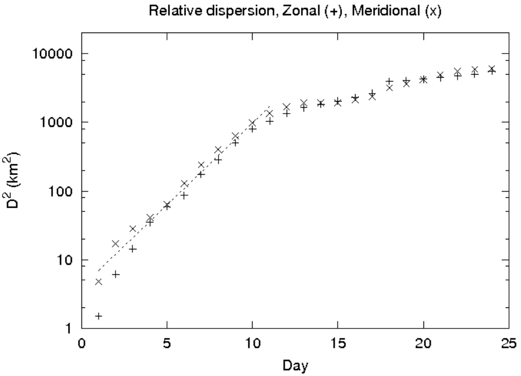


SCULP drifters: Gulf of Mexico



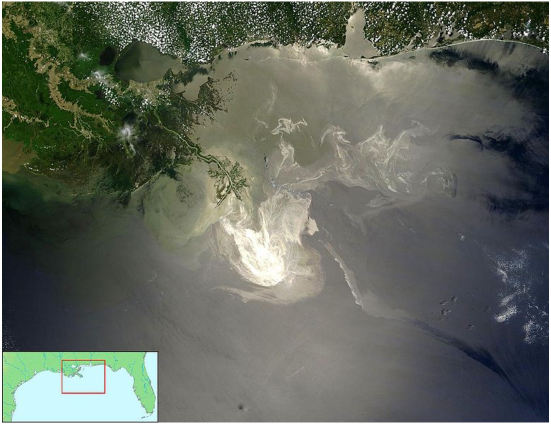
140 "chance pairs" with $r_0 = 1km$

SCULP drifters: Gulf of Mexico



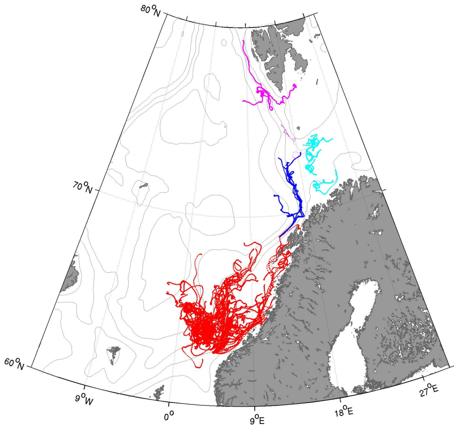
LaCasce and Ohlmann (2003)

Application: Gulf of Mexico



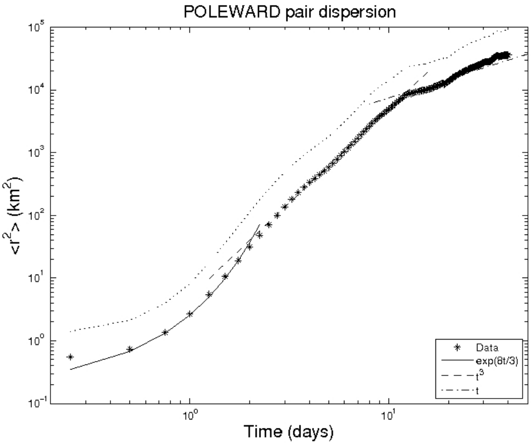
National Geographic, April 26, 2010

POLEWARD drifters: Nordic Seas



93 "original pairs" with $r_0 = 2km$

POLEWARD drifters: Nordic Seas



Koszalka, LaCasce, Orvik (2009)

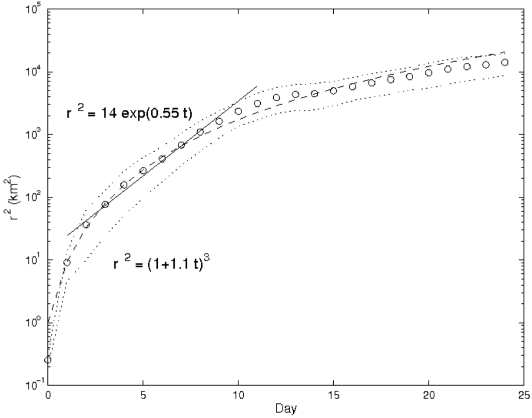
EOLE reanalysis

But grumblings in the audience...

• Using the "finite scale Lyapunov exponent" (FSLE), Lacorata et al. (2003) suggested that the EOLE dispersion was not exponential but in line with Richardson ($K_2 \propto r^{4/3}$)

• With the FSLE, Lumpkin and Ellipot (2010) found only Richardson dispersion for drifters in the western Atlantic

SCULP dispersion



PDFs

Previous studies based primarily on dispersion, the second moment of the displacements

The FSLE is similarly a single average number (time) for drifters

Get more information from the probability density function (PDF)

PDFs

Richardson (1926) proposed that smoke dispersion could be modelled using:

$$\frac{\partial}{\partial t} C = \nabla \cdot (K_2(r) \nabla C)$$

Cloud dispersion is equivalent to pair dispersion. So the PDF of the pair separations, $p(r, t)$, obeys a Fokker-Plank (FP) equation:

$$\frac{\partial}{\partial t} p = \nabla \cdot (K_2(r) \nabla p)$$

- Can solve the FP equation for the inertial ranges in 2D

Two-dimensional turbulence

- Inverse Energy cascade

$$E(\kappa) \propto \kappa^{-5/3}, \quad \kappa = \epsilon^{1/3} r^{4/3}$$

Asymptotic solution:

$$p(r,t) = (\frac{3}{2})^5 \frac{1}{2\epsilon t^3}\, exp(-\frac{9r^{2/3}}{4\epsilon^{1/3} t})$$

Dispersion: $< r^2 >= 5.2675\, \epsilon\, t^3$

Richardson, 1926; Boffetta and Celani, 2000; LaCasce, 2010

Two-dimensional turbulence

- Enstrophy cascade

$$E(\kappa) \propto \kappa^{-3}, \quad \kappa = \eta^{1/3} r^2$$

Solution:

$$p(r,t) = \frac{1}{2(\pi \eta^{1/3} t)^{1/2}\, r_0^2}\, exp(-\frac{[ln(r/r_0) + 2\eta^{1/3} t]^2}{4\eta^{1/3} t})$$

Dispersion: $< r^2 >= r_0^2 exp(8\eta^{1/3} t)$

Lundgren, 1981; Bennett, 2006; LaCasce, 2010

Large scales

- Uncorrelated motion

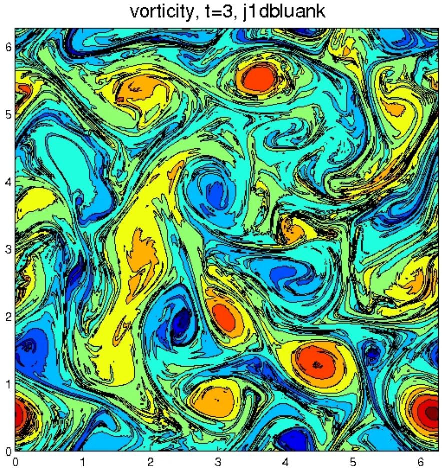
$$\kappa = const.$$

Asymptotic solution:

$$p(r,t) = \frac{1}{\kappa t}\, exp(-\frac{r^2}{2\kappa t})$$

Dispersion: $< r^2 >= 2\kappa t$

2-D turbulence model



2-D turbulence model

Solves the vorticity equation:

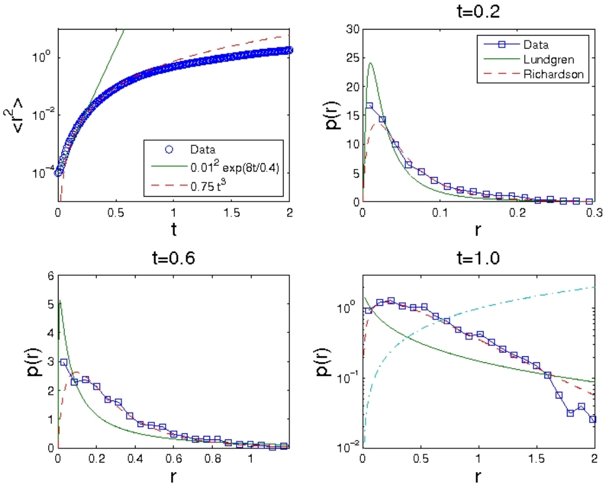
$$\frac{\partial}{\partial t}\zeta + J(\psi, \zeta) = \mathcal{F} + \mathcal{D} \qquad (1)$$

1) Energy cascade: $\kappa_F = [100, 120]$
$\mathcal{D} = \nu \nabla^{-2} \zeta$, exponential cut-off filter

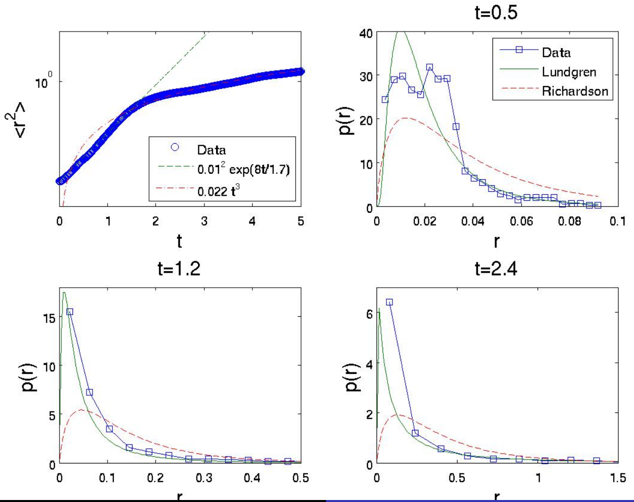
2) Enstrophy cascade: $\kappa_F = [1, 5]$
$\mathcal{D} \rightarrow$ exponential filter

Doubly-periodic, 512^2 grid points, 2000 particles

Energy cascade

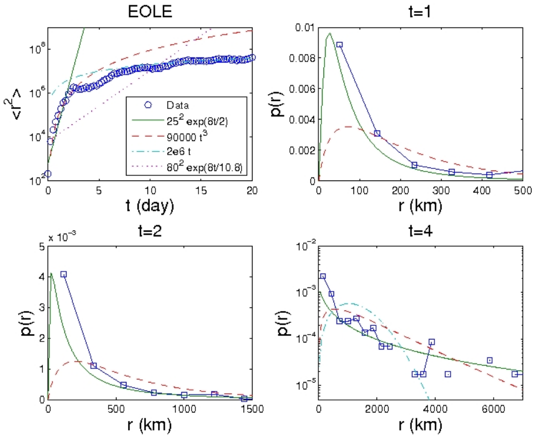


Enstrophy cascade



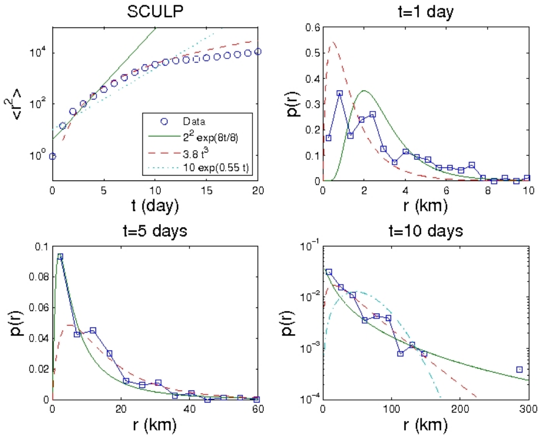
EOLE balloons

426 pairs at 200 mb, $r_0 = 25km$



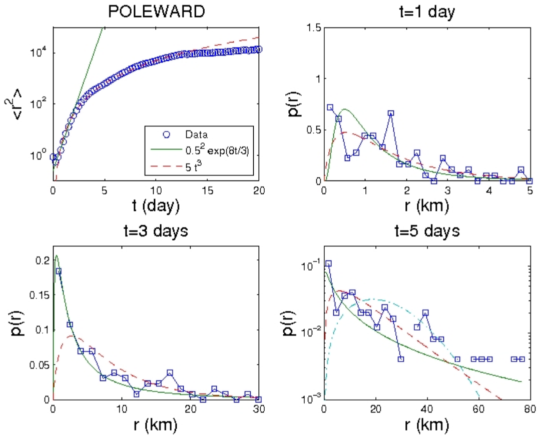
SCULP drifters

188 pairs, $r_0 = 2km$



POLEWARD drifters, Nordic Seas

93 pairs, $r_0 = 2km$



Summary, Lecture III

- Relative dispersion concerns pairs of particles and reflects the spread of a cloud of tracer
- For homogeneous flows, relative dispersion depends on the Eulerian energy spectrum
- Observations support *exponential growth* below the deformation radius in the lower stratosphere and at the ocean surface
- Implies the sub-deformation scale energy spectrum is at least as steep as κ^{-3}
- Important for parameterizations and for simulating tracer spreading, e.g. oil and volcanic ash

References

Bennett, A.F. (1984): Relative dispersion: local and nonlocal dynamics. J. Atmos. Sci. (41), 1881-1886.

Davis, R. E. (1991): Observing the general circulation with floats. Deep Sea Res. (38), S531-S571.

Einstein, A. (1905): On the movement of small particles suspended in stationary liquids required by the molecular-kinetic theory of heat. Ann. Physik (17), 549-560.

Ferrari, R. and Nikurashin, M. (2010): Suppression of eddy diffusivity across jets in the Southern Ocean. J. Phys. Oceanogr. (40), 1501-1519.

Klocker, A., R. Ferrari and J. H. LaCasce (2012a): Estimating suppression of eddy mixing by mean flows. J. Phys. Oceanogr. (42), 1566-1576.

Klocker, A., R. Ferrari, J. H. LaCasce and S. T. Merrifield (2012b): Reconciling Lagrangian and Eulerian estimates of eddy diffusivities: tracer and particle-based estimates. J. Mar. Res. (in press).

LaCasce, J.H. and C. Ohlmann (2003): Relative dispersion at the surface of the Gulf of Mexico. J. Mar. Res. (61), 285-312.

LaCasce, J. H. (2008): Statistics from Lagrangian observations . Prog. Oceanogr. (77), 1-29.

LaCasce, J. H. (2010): Relative displacement PDFs from balloons and drifters. J. Mar. Res. (68), 433-457.

References

Lundgren, T.S. (1981): Turbulent pair dispersion and scalar diffusion. J. Fluid Mech. (111), 27-57.
Lumpkin, R., A.-M. Treguier and K. Speer (2002): Lagrangian eddy scales in the northern Atlantic Ocean. J. Phys. Oceanogr. (32), 2425-2440.
Richardson, L.F. (1926): Atmospheric diffusion on a distance-neighbour graph. Proc. Royal Soc. London A (110), 709-737.
Taylor, G. I. (1921): Diffusion by continuous moments. Proc. London Math. Soc., (20), 196-211.