## The vertical gradient of potential density

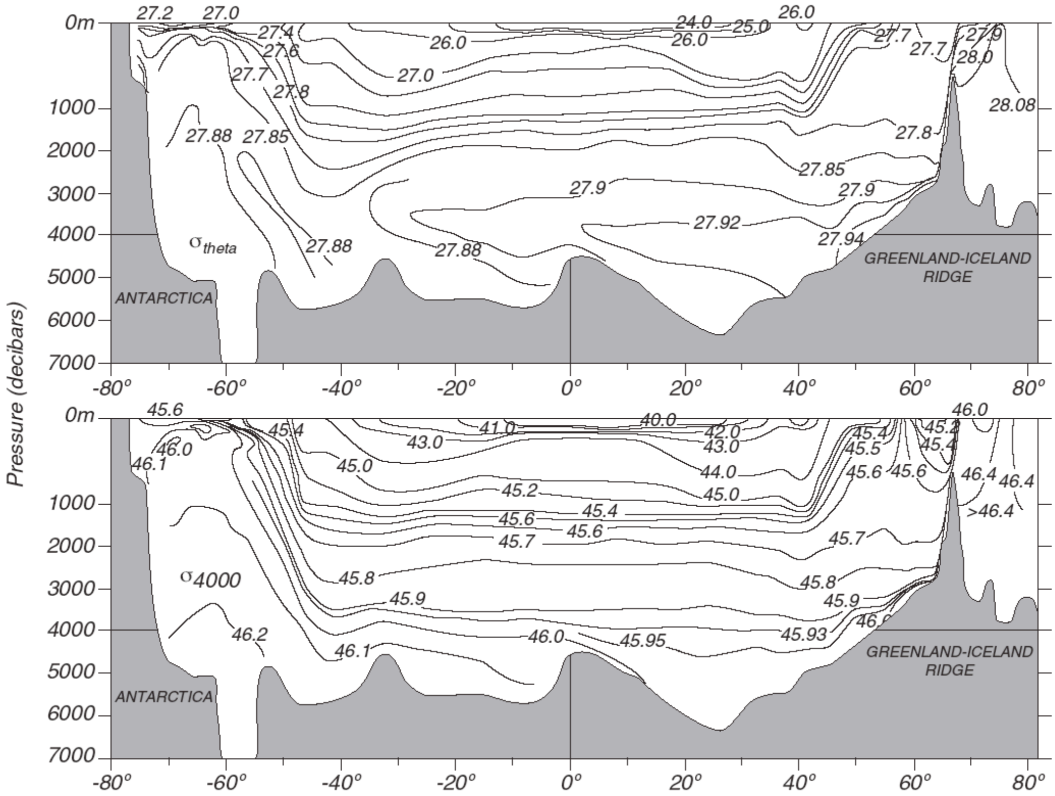



Figure 6.10 Vertical sections of density in the western Atlantic. Note that the depth scale changes at 1000 m depth. **Upper**: $\sigma_\Theta$, showing an apparent density inversion below 3,000 m. **Lower**: $\sigma_4$ showing continuous increase in density with depth. After Lynn and Reid (1968).

The potential density of a seawater sample $(S_A, \Theta, p)$, referenced to reference pressure $p_r$ is given by $\rho^\Theta(S_A, \Theta) = \hat{\rho}(S_A, \Theta, p_r)$. The vertical gradient of the natural logarithm of potential density is $\beta^\Theta(p_r)$ times the vertical gradient of Absolute Salinity minus $\alpha^\Theta(p_r)$ times the vertical gradient of Conservative Temperature,

$$\frac{1}{\rho^\Theta}\frac{\partial \rho^\Theta}{\partial z} = \beta^\Theta(p_r) S_{A_z} - \alpha^\Theta(p_r)\Theta_z . \qquad \text{(A.26.2)}$$

The ratio of this vertical gradient of potential density to the square of the buoyancy frequency is given by (Tutorial exercise)

$$\frac{-g\rho^{-1}\rho_z^\Theta}{N^2} = \frac{\beta^\Theta(p_r)}{\beta^\Theta(p)}\frac{\left[R_\rho/r-1\right]}{\left[R_\rho-1\right]} = \frac{\beta^\Theta(p_r)}{\beta^\Theta(p)}\frac{1}{G^\Theta} \approx \frac{1}{G^\Theta}, \qquad \text{(3.20.5)}$$

where $r$ is the ratio of the slope on the $S_A - \Theta$ diagram of an isoline of potential density with reference pressure $p_r$ to the slope of a potential density surface with reference pressure $p$, and is defined by

$$r = \frac{\alpha^\Theta(S_A,\Theta,p)/\beta^\Theta(S_A,\Theta,p)}{\alpha^\Theta(S_A,\Theta,p_r)/\beta^\Theta(S_A,\Theta,p_r)}, \qquad \text{(3.17.2)}$$

and the "isopycnal temperature gradient ratio" $G^\Theta$ is defined by

$$G^\Theta \equiv \frac{\left[R_\rho-1\right]}{\left[R_\rho/r-1\right]} \quad \text{where} \quad R_\rho = \frac{\alpha^\Theta\Theta_z}{\beta^\Theta(S_A)_z} \qquad \text{(3.17.4)}$$

is the ratio of the vertical contribution from Conservative Temperature to that from Absolute Salinity to the static stability $N^2$ of the water column. The name "isopycnal temperature gradient ratio" is chosen for $G^\Theta$ because it can be shown that $G^{\Theta}$ is the ratio of the gradient of Conservative Temperature in a potential density surface to that in a neutral tangent plane (Tutorial exercise),

$$\nabla_{\sigma}\Theta \;=\; G^{\Theta}\nabla_{n}\Theta \;. \qquad (3.17.3)$$

The saline contraction coefficient $\beta^{\Theta}\left(S_{\mathrm{A}},\Theta,p\right)$ does not vary very much from a constant value compared with variation of the thermal expansion coefficient $\alpha^{\Theta}\left(S_{\mathrm{A}},\Theta,p\right)$. That is, you make a 10% - 20% error by approximating $r$ as

$$r \;\approx\; \frac{\alpha^{\Theta}\left(S_{\mathrm{A}},\Theta,p\right)}{\alpha^{\Theta}\left(S_{\mathrm{A}},\Theta,p_{\mathrm{r}}\right)} \;. \qquad (3.17.2\_approx)$$

There is never any reason to actually make this approximation in numerical work, rather this approximation can aid in thinking about what causes what in the ocean. [You can check that this is a good approximation by inspection of the red and blue potential density contours on the above $S_{\mathrm{A}}-\Theta$ diagram.]

Also, the slope difference between that of a neutral tangent plane and a potential density surface is given by (Tutorial exercise)

$$\begin{aligned}\nabla_{n}z-\nabla_{\sigma}z \;&=\; \frac{\nabla_{n}\Theta-\nabla_{\sigma}\Theta}{\Theta_{z}} \;=\; \left(1-G^{\Theta}\right)\frac{\nabla_{n}\Theta}{\Theta_{z}}\\ &=\; \frac{R_{\rho}\left[1-r\right]}{\left[R_{\rho}-r\right]}\frac{\nabla_{n}\Theta}{\Theta_{z}} \;=\; \frac{R_{\rho}\left[1-r\right]}{r\left[R_{\rho}-1\right]}\frac{\nabla_{\sigma}\Theta}{\Theta_{z}}.\end{aligned} \qquad (3.18.1)$$

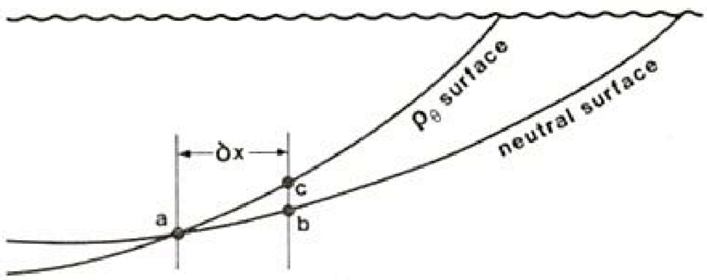



FIG. 1. Sketch of a cross section through the ocean showing a neutral surface and a potential density surface passing through point a. At a horizontal distance $\delta x$ from point a, a vertical cast cuts the two surfaces at points b and c.

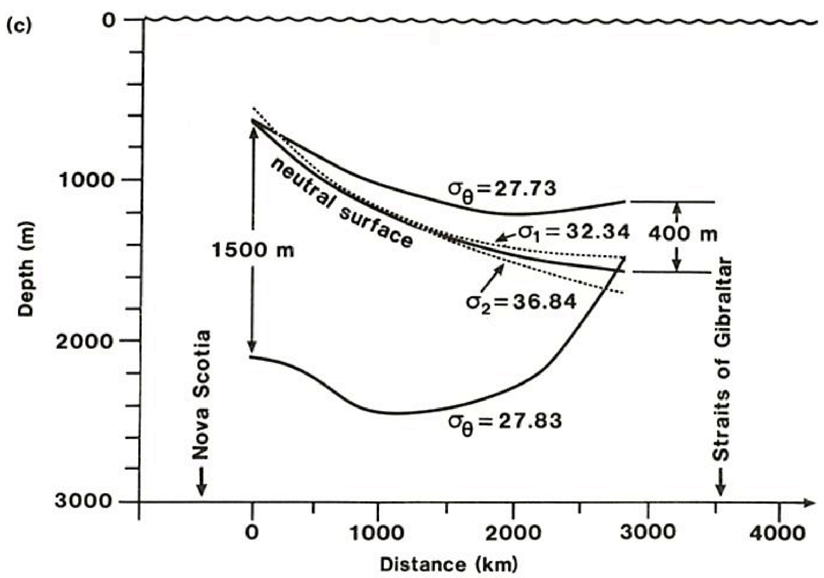
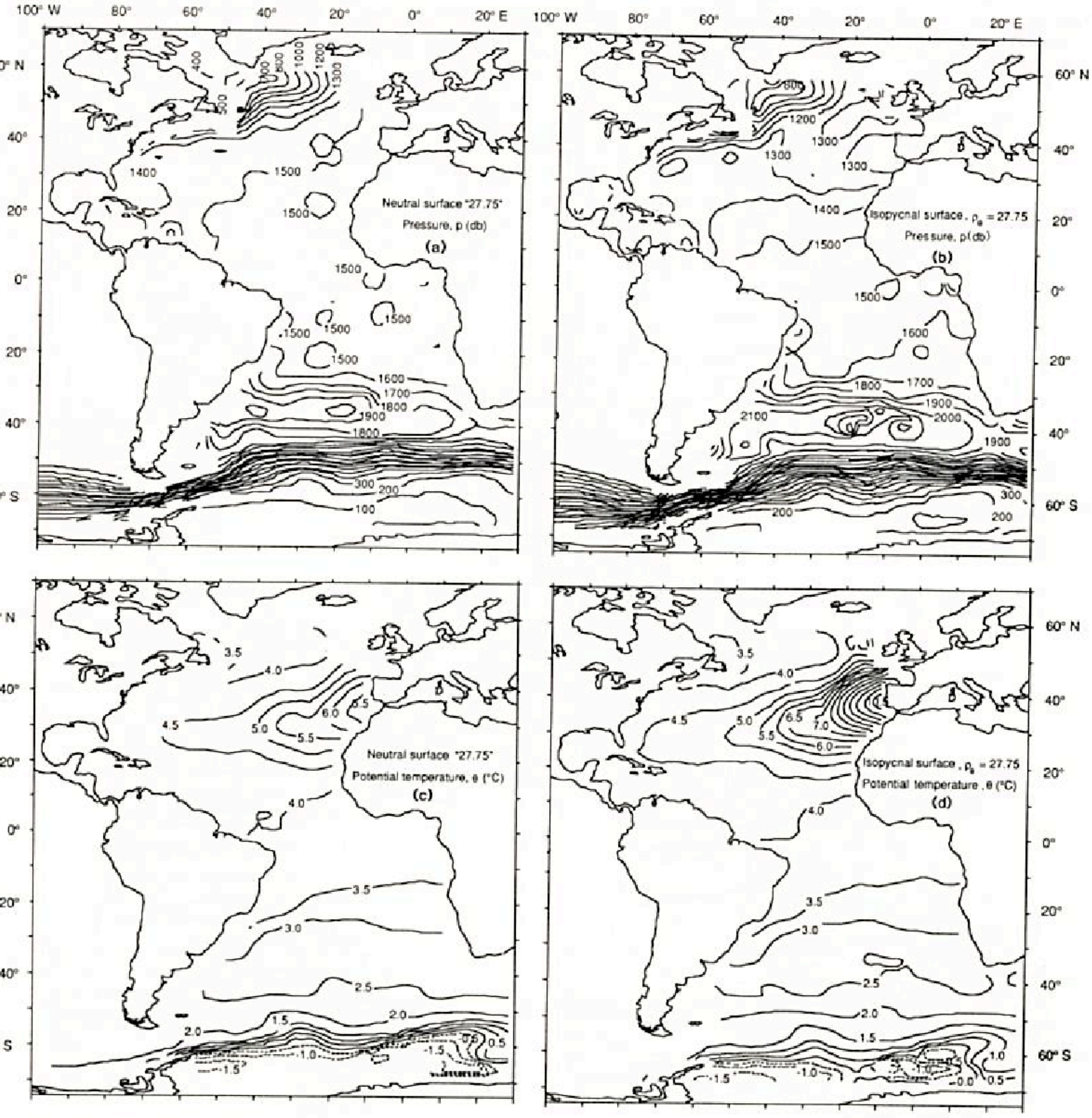



FIG. 7. Maps of pressure on two potential density surfaces: (a) $\sigma_\theta$ = 27.73; (b) $\sigma_\theta$ = 27.83. The potential density surfaces intersect the same neutral surface (NSa of Fig. 6) at different positions. This is illustrated in cross section in (c), which goes from near Nova Scotia on the left to near the Straits of Gibraltar on the right. Also shown (dashed lines) are a potential density surface referenced to a pressure of 1000 db ($\sigma_1$ = 32.34) and a potential density surface referenced to 2000 db ($\sigma_2$ = 36.84).



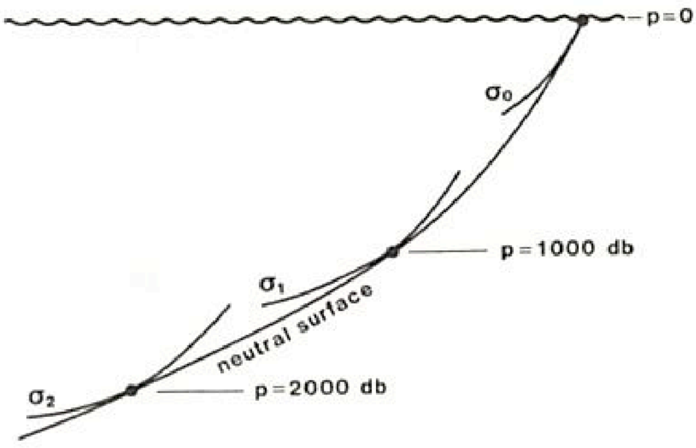
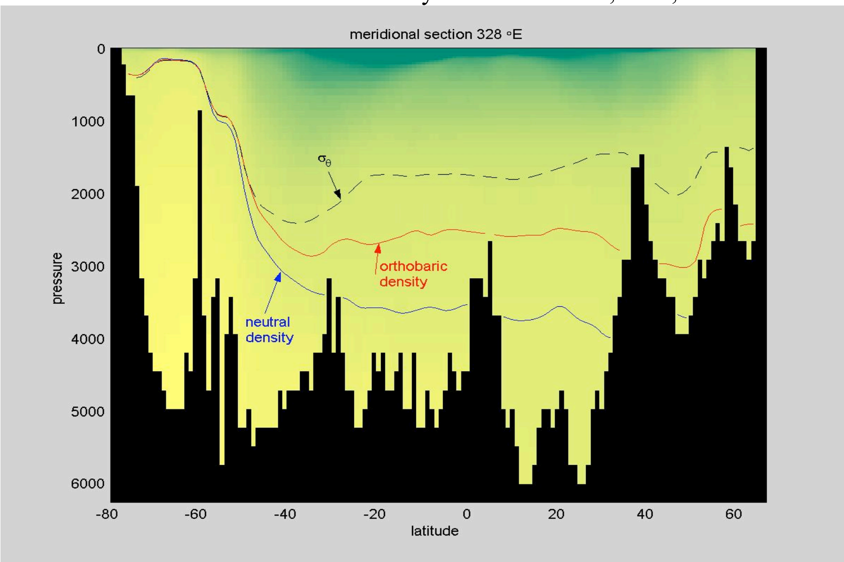
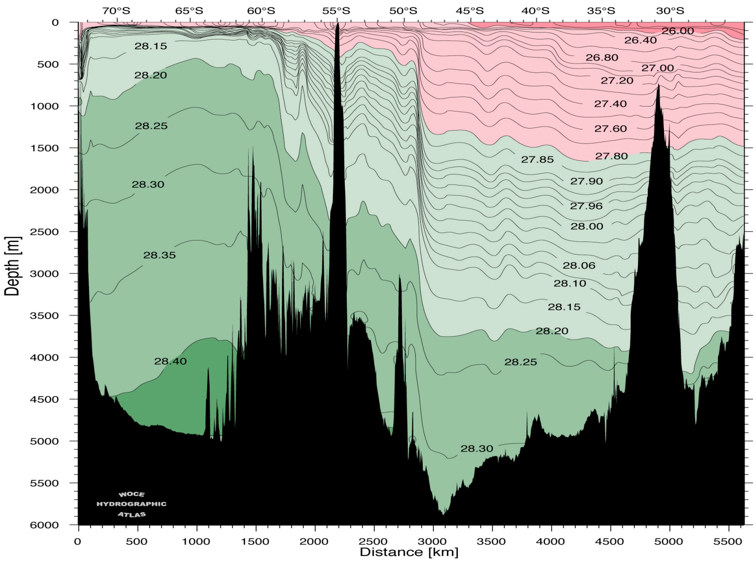
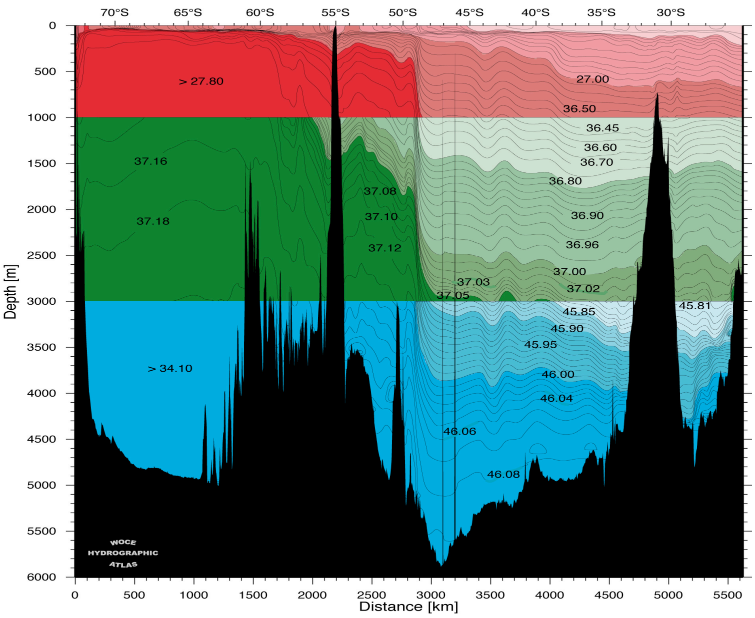




FIG. 2. Sketch of a neutral surface and three different potential density surfaces, referenced to 0 db, 1000 db and 2000 db. The neutral surface is tangential to potential density surfaces only at the reference pressure of those potential density surfaces. In this way, the neutral surface can be regarded as the envelope curve of many locally referenced potential density surfaces with continually changing reference pressures. The definition of a neutral surface adopted in this paper avoids the concept of potential density and in particular, avoids the changing reference pressure which is endemic to a neutral surface defined in terms of potential density concepts.




Below is a cross-section of Neutral Density in the Southern Ocean.




Before Neutral Density was available, cross-sections of density used potential density referenced to three different reference pressures, 0 dbar, 2000 dbar, and 4000 dbar, as shown above.

## Geostrophic, hydrostatic and "thermal wind" equations

The geostrophic approximation to the horizontal momentum equations (Eqn. (B9)) equates the Coriolis term to the horizontal pressure gradient $\nabla_z P$ so that the geostrophic equation is

$$f\,\mathbf{k}\times\rho\mathbf{u} \;=\; -\nabla_z P \qquad \text{or} \qquad f\mathbf{v} \;=\; \tfrac{1}{\rho}\,\mathbf{k}\times\nabla_z P \;=\; g\,\mathbf{k}\times\nabla_p z\,, \qquad (3.12.1)$$

where $\mathbf{u}$ is the three dimensional velocity and $\mathbf{v} = -\mathbf{k}\times(\mathbf{k}\times\mathbf{u})$ is the horizontal velocity where $\mathbf{k}$ is the vertical unit vector (pointing upwards) and $f$ is the Coriolis parameter. The last part of the above equation has used $\nabla_z P = -P_z \nabla_p z$ from Eqn. (3.12.4b) below and the hydrostatic approximation, which is the following approximation to the vertical momentum equation (B9),

$$P_z = -g\rho. \qquad (3.12.2)$$

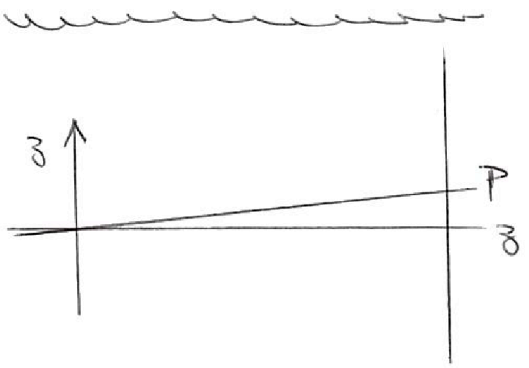
The use of $P$ in these equations rather than $p$ serves to remind us that in order to retain the usual units for height, density and the gravitational acceleration, pressure in these dynamical equations must be expressed in Pa not dbar.

The so called "thermal wind" equation is an equation for the vertical gradient of the horizontal velocity under the geostrophic approximation. Vertically differentiating Eqn. (3.12.1) and using the hydrostatic equation Eqn. (3.12.2), the thermal wind can be written

$$\boxed{f\,\mathbf{v}_z \;=\; \left(\tfrac{1}{\rho}\right)_z \mathbf{k}\times\nabla_z P + \tfrac{1}{\rho}\mathbf{k}\times\nabla_z\left(P_z\right) \;=\; -\tfrac{g}{\rho}\,\mathbf{k}\times\nabla_p\rho \;=\; \tfrac{N^2}{g\rho}\,\mathbf{k}\times\nabla_n P,} \qquad (3.12.3)$$

where $\nabla_p$ is the projected lateral gradient operator in the isobaric surface (see Eqn. (3.11.3)). The last part of this equation relates the "thermal wind", $f\,\mathbf{v}_z$, to the pressure gradient in the neutral tangent plane. Note that the Boussinesq approximation has not been made to derive any part of Eqn. (3.12.3). Under the Boussinesq approximation, $\nabla_p\rho$ is approximated by $\nabla_z\rho$, and the last term in Eqn. (3.12.3) is approximated as $-N^2\,\mathbf{k}\times\nabla_n z$. The derivation of Eqn. (3.12.3) proceeds as follows. To go from the second part of Eqn. (3.12.3) to the third part use is made of

$$\nabla_p\rho \;=\; \nabla_z\rho \;+\; \rho_z\nabla_p z \qquad \text{and} \qquad \nabla_p P \;=\; \mathbf{0} \;=\; \nabla_z P \;+\; P_z\nabla_p z\,. \qquad (3.12.4a,b)$$




To go from the third part of Eqn. (3.12.3) to the final part, use is made of Eqn. (3.12.4a) and $\nabla_n\rho \;=\; \nabla_z\rho \;+\; \rho_z\nabla_n z$, which, when combined gives

$$\nabla_p\rho \;=\; \nabla_n\rho \;-\; \rho_z\left(\nabla_n z - \nabla_p z\right). \qquad (3.12.5)$$

Now Eqn. (3.12.4b) is used together with $\nabla_n P \;=\; \nabla_z P \;+\; P_z\nabla_n z$ to find

$$\nabla_n P \;=\; P_z\left(\nabla_n z - \nabla_p z\right), \qquad (3.12.6)$$

and this is substituted into Eqn. (3.12.5) to find

$$\nabla_p\rho \;=\; \nabla_n\rho \;-\; \rho_z\nabla_n P/P_z\,. \qquad (3.12.7)$$

Now along a neutral tangent plane we know that $\nabla_n\rho \;=\; \rho\kappa\nabla_n P$ ($\kappa$ is the isentropic and isohaline compressibility of seawater) and substituting this into

Eqn. (3.12.7) leads to the final expression of Eqn. (3.12.3), namely $\frac{N^2}{g\rho}\,\mathbf{k}\times\nabla_n P$ (recognizing that the buoyancy frequency is defined by $N^2 = g\left(\kappa P_z - \frac{1}{\rho}\rho_z\right)$).

## Neutral helicity

From page 94 of these lecture notes we know that the normal $\mathbf{n}$ to the neutral tangent plane is given by

$$g^{-1} N^2 \mathbf{n} = -\rho^{-1}\nabla\rho + \kappa\nabla P = -\rho^{-1}\left(\nabla\rho - \nabla P / c^2\right) \\ = \alpha^{\Theta}\nabla\Theta - \beta^{\Theta}\nabla S_{\mathrm{A}}. \qquad (3.11.1)$$

It is natural to think that all these little tangent planes would link up and form a well-defined surface, but this is not actually the case in the ocean. In order to understand why the ocean chooses to be so ornery [bad-tempered] we need to understand what property the normal $\mathbf{n}$ to a surface must fulfill in order that the surface exists. We will find that this property is that the scalar product of the normal of the surface $\mathbf{n}$ and the curl of $\mathbf{n}$ must be zero everywhere on the surface; that is $\mathbf{n}\cdot\nabla\times\mathbf{n}$ must be zero everywhere on the surface.

In general, for a surface to exist in $(x,y,z)$ space there must be a function $\phi(x,y,z)$ that is constant on the surface and whose gradient $\nabla\phi$ is in the direction of the normal to the surface, $\mathbf{n}$. That is, there must be an integrating factor $b(x,y,z)$ such that $\nabla\phi = b\mathbf{n}$. Assuming now that the surface does exist, consider a line integral of $b\mathbf{n}$ along a closed curved path in the surface. Since the line element of the integration path is everywhere normal to $\mathbf{n}$, the closed line integral is zero, and by Stokes's theorem, the area integral of $\nabla\times(b\mathbf{n})$ must be zero over the area enclosed by the closed curved path. Since the area element of integration $d\mathbf{A}$ is in the direction $\mathbf{n}$, it is clear that $\nabla\times(b\mathbf{n})\cdot d\mathbf{A}$ is proportional to $\nabla\times(b\mathbf{n})\cdot\mathbf{n}$. The only way that this area integral can be guaranteed to be zero for all such closed paths is if the integrand is zero everywhere on the surface, that is, if $\nabla\times(b\mathbf{n})\cdot\mathbf{n} = (\nabla b\times\mathbf{n})\cdot\mathbf{n} + b(\nabla\times\mathbf{n})\cdot\mathbf{n} = 0$, that is, if $\mathbf{n}\cdot\nabla\times\mathbf{n} = 0$ at all locations on the surface.

For the case in hand, the normal to the neutral tangent plane is in the direction $\alpha^{\Theta}\nabla\Theta - \beta^{\Theta}\nabla S_{\mathrm{A}}$ and we define the neutral helicity $H^{\mathrm{n}}$ as the scalar product of $\alpha^{\Theta}\nabla\Theta - \beta^{\Theta}\nabla S_{\mathrm{A}}$ with its curl,

$$H^{\mathrm{n}} \equiv \left(\alpha^{\Theta}\nabla\Theta - \beta^{\Theta}\nabla S_{\mathrm{A}}\right)\cdot\nabla\times\left(\alpha^{\Theta}\nabla\Theta - \beta^{\Theta}\nabla S_{\mathrm{A}}\right). \qquad (3.13.1)$$

Neutral tangent planes (which do exist) do not link up in space to form a well-defined neutral surface unless the neutral helicity $H^{\mathrm{n}}$ is everywhere zero on the surface.

Recognizing that both the thermal expansion coefficient and the saline contraction coefficient are functions of $(S_{\mathrm{A}},\Theta,p)$, neutral helicity $H^{\mathrm{n}}$ may be expressed as the following four expressions, all of which are proportional to the thermobaric coefficient $T_{\mathrm{b}}^{\Theta}$ of the equation of state,

$$\begin{aligned} H^{\mathrm{n}} &= \beta^{\Theta} T_{\mathrm{b}}^{\Theta}\nabla P\cdot\nabla S_{\mathrm{A}}\times\nabla\Theta \\ &= P_z\,\beta^{\Theta} T_{\mathrm{b}}^{\Theta}\left(\nabla_p S_{\mathrm{A}}\times\nabla_p\Theta\right)\cdot\mathbf{k} \\ &= g^{-1} N^2 T_{\mathrm{b}}^{\Theta}\left(\nabla_n P\times\nabla_n\Theta\right)\cdot\mathbf{k} \\ &\approx g^{-1} N^2 T_{\mathrm{b}}^{\Theta}\left(\nabla_a P\times\nabla_a\Theta\right)\cdot\mathbf{k} \end{aligned} \qquad (3.13.2)$$

where $P_z$ is simply the vertical gradient of pressure ($\mathrm{Pa\ m^{-1}}$) and $\nabla_n\Theta$ and $\nabla_p\Theta$ are the two-dimensional gradients of $\Theta$ in the neural tangent plane and in the horizontal plane (actually the isobaric surface) respectively. The gradients $\nabla_a P$ and $\nabla_a\Theta$ are taken in an approximately neutral surface. Neutral helicity has units of $\mathrm{m^{-3}}$. Recall that the thermobaric coefficient is given by

$$T_{\mathrm{b}}^{\Theta} = \beta^{\Theta}\left(\alpha^{\Theta}/\beta^{\Theta}\right)_P = \alpha_P^{\Theta} - \left(\alpha^{\Theta}/\beta^{\Theta}\right)\beta_P^{\Theta}. \qquad (3.8.2)$$

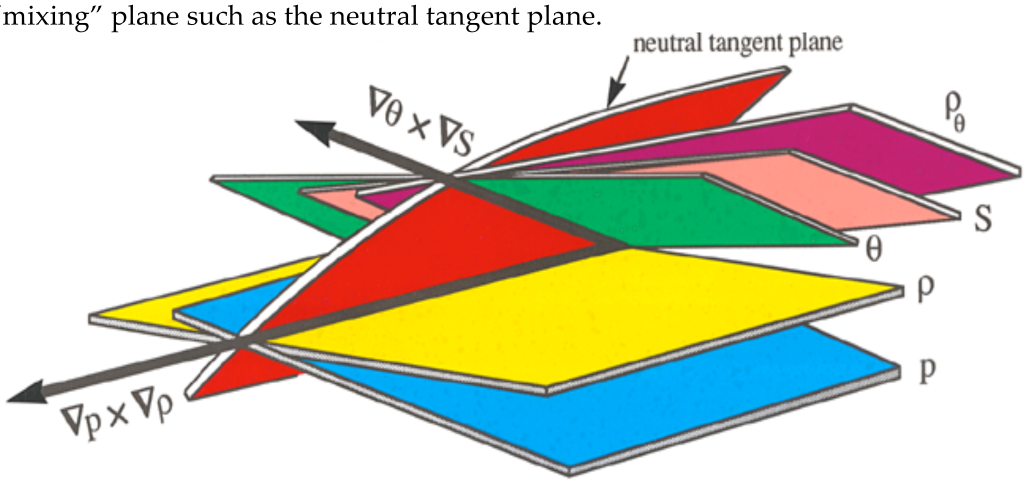
## The geometrical interpretation of neutral helicity

How can we understand neutral helicity $H^{\mathrm{n}}$ geometrically? Recall the definition of a neutral tangent plane, Eqn. (3.11.2), namely

$$-\rho^{-1}\nabla_n\rho + \kappa\nabla_n P \;=\; \alpha^{\Theta}\nabla_n\Theta - \beta^{\Theta}\nabla_n S_{\mathrm{A}} \;=\; \mathbf{0}. \qquad (3.11.2)$$

This implies that the two lines $\nabla P\times\nabla\rho$ and $\nabla\Theta\times\nabla S_{\mathrm{A}}$ both lie in the neutral tangent plane. This is because along the line $\nabla P\times\nabla\rho$ both pressure and *in situ* density are constant, and along this line the neutral property is satisfied. Similarly, along the line $\nabla\Theta\times\nabla S_{\mathrm{A}}$ both Conservative Temperature and Absolute Salinity are constant, which certainly describes a line in the neutral tangent plane. Hence the picture emerges below of the geometry in $(x,y,z)$ space of six planes, intersecting in one of the two lines $\nabla P\times\nabla\rho$ and $\nabla\Theta\times\nabla S_{\mathrm{A}}$. The neutral tangent plane is the only plane that includes both of these desirable lines.

Why are these lines "desirable"? Well $\nabla P\times\nabla\rho$ is desirable because it is the direction of the "thermal wind", and $\nabla\Theta\times\nabla S_{\mathrm{A}}$ is desirable because adiabatic and isohaline motion occurs along this line; a necessary attribute of a well-bred "mixing" plane such as the neutral tangent plane.



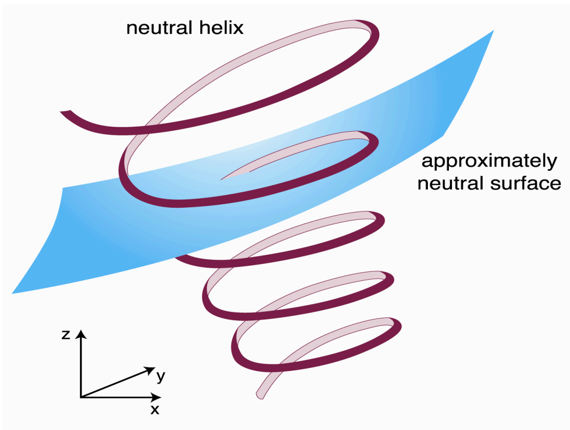

Prolonged gazing at the above figure while examining the definition of neutral helicity, $H^{\mathrm{n}}$, Eqn. (3.13.2), shows that neutral helicity vanishes when the two vectors $\nabla P\times\nabla\rho$ and $\nabla\Theta\times\nabla S_{\mathrm{A}}$ coincide, and that this occurs when the two-dimensional gradients $\nabla_n\Theta$ are $\nabla_n P$ parallel.

Neutral helicity is proportional to the component of the vertical shear of the geostrophic velocity ($\mathbf{v}_z$, the "thermal wind") in the direction of the temperature gradient along the neutral tangent plane $\nabla_n\Theta$, since, from Eqn. (3.12.3) and the third line of (3.13.2) we find that

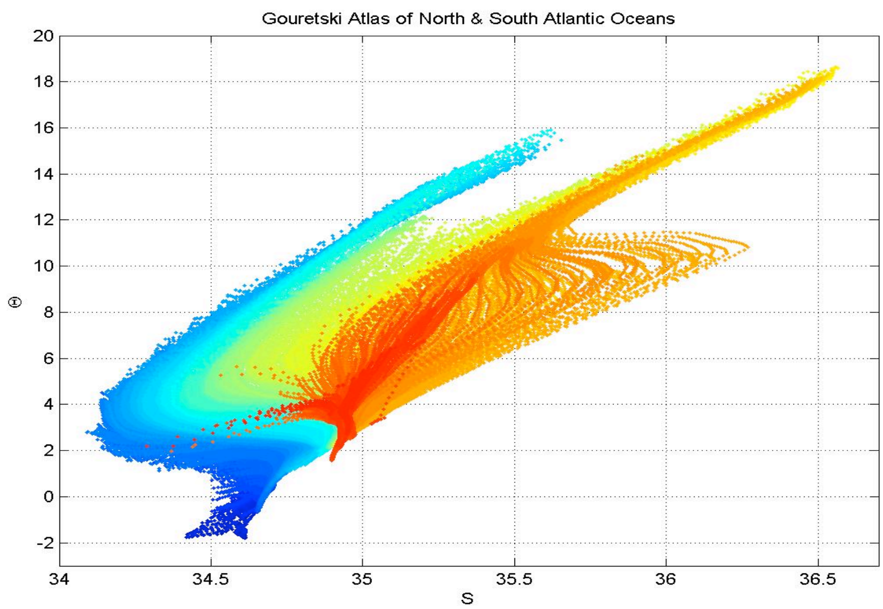
$$H^{\mathrm{n}} = \rho T_{\mathrm{b}}^{\Theta} f\mathbf{v}_z\cdot\nabla_n\Theta\,. \qquad (3.13.3)$$

Interestingly, for given magnitudes of the epineutral gradients of pressure and Conservative Temperature, neutral helicity is maximized when these gradients are perpendicular since neutral helicity is proportional to $T_{\mathrm{b}}^{\Theta}\left(\nabla_n P\times\nabla_n\Theta\right)\cdot\mathbf{k}$ (see Eqn. (3.13.2)), while the dianeutral advection of thermobaricity, $e^{\mathrm{Tb}} = -gN^{-2}K\,T_{\mathrm{b}}^{\Theta}\nabla_n\Theta\cdot\nabla_n P$, is maximized when $\nabla_n\Theta$ and $\nabla_n P$ are parallel (see Eqn. **(A.22.4)**).

Because of the non-zero neutral helicity, $H^{n}$, in the ocean, lateral motion following neutral tangent planes has the character of helical motion. That is, if we ignore the effects of diapycnal mixing processes (as well as ignoring cabbeling and thermobaricity), the mean flow around ocean gyres still passes through any well-defined "density" surface because of the helical nature of neutral trajectories, caused in turn by the non-zero neutral helicity. We will return to this mean vertical motion caused by the ill-defined nature of "neutral surfaces" in a few pages.

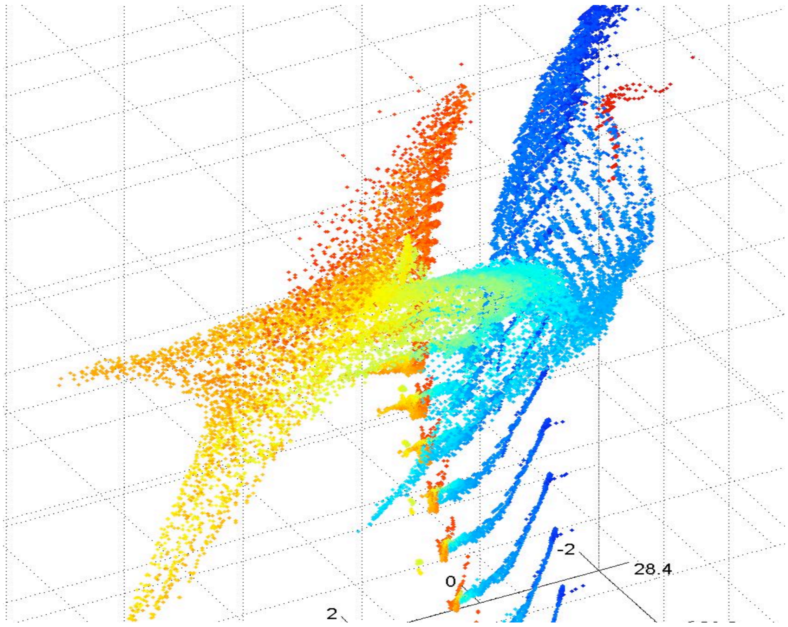



## The skinny nature of the ocean; why is the ocean 95% empty?




The above diagram contains all of the ocean hydrography below 200 dbar from both the North and South Atlantic ocean. The colour represents the latitude, with blue in the south, red in the north and green in the equatorial region. It is seen that the data fill the area on this $S_{\mathrm{A}} - \Theta$ diagram, leaving no holes.

When considering the plotting of this same data on a three-dimensional $S_{\mathrm{A}} - \Theta - p$ "plot", one could be forgiven for thinking that the data would fill in a solid shape in these three dimensions. But this is not observed. Rather than the $S_{\mathrm{A}} - \Theta - p$ data occupying the volume inside, say, a packet of Toblerone chocolate, instead, the data resides on the cardboard of the Toblerone packet and the chocolate is missing.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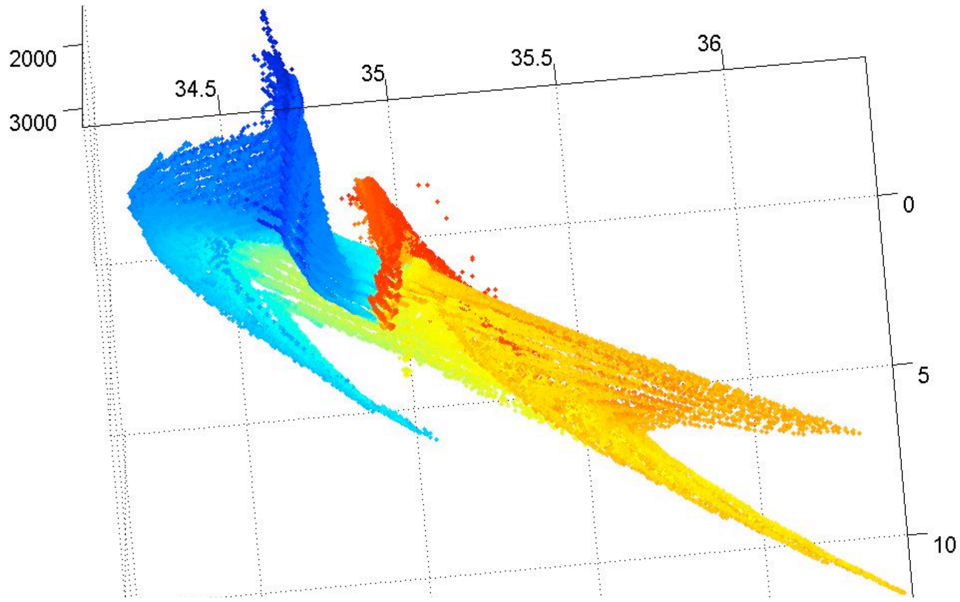
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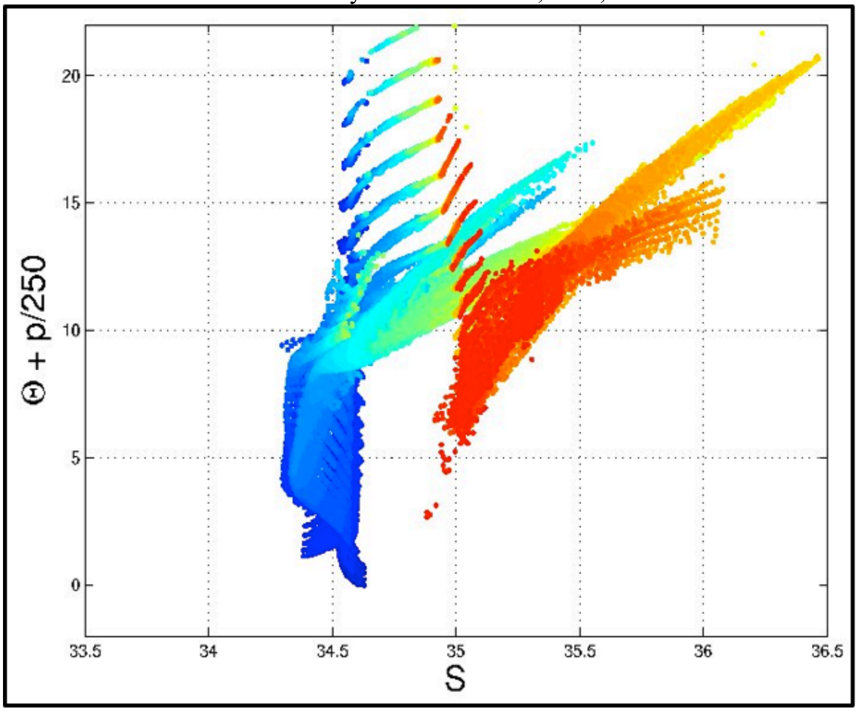
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## The skinny nature of the ocean; implication for neutral helicity

If all the $(S_A, \Theta, p)$ data from the whole global ocean were to lie exactly on a single surface in $(S_A, \Theta, p)$ space, we will prove that this requires $\nabla S_A \times \nabla\Theta \cdot \nabla P = 0$ everywhere in physical $(x, y, z)$ space. That is, we will prove that the skinniness of the ocean hydrography in $(S_A, \Theta, p)$ space is a direct indication of the smallness of neutral helicity $H^n$.

Since, under our assumption, all the $(S_A, \Theta, p)$ data from the whole global ocean lies on the single surface in $(S_A, \Theta, p)$ space we have

$$f\left(S_A, \Theta, p\right) = 0 \qquad \text{(Twiggy\_01)}$$

for every $(S_A, \Theta, p)$ observation drawn for the whole global ocean in physical $(x, y, z)$ space. Taking the spatial gradient of this equation in physical $(x, y, z)$ space we have $\nabla f = 0$ since $f$ is zero at every point in physical $(x, y, z)$ space. Expanding $\nabla f$ in terms of the spatial gradients $\nabla S_A$, $\nabla\Theta$, and $\nabla P$, and taking the scalar product with $\nabla S_A \times \nabla\Theta$ we find that

$$\left.\frac{\partial f}{\partial P}\right|_{S_A,\Theta} \nabla P \cdot \nabla S_A \times \nabla\Theta = 0 . \qquad \text{(Twiggy\_02)}$$

In the general case of $f_P \neq 0$, the result $\nabla P \cdot \nabla S_A \times \nabla\Theta = 0$ is proven. In the special case $f_P = 0$, $f$ is independent of $P$ so that we have a simpler equation for the surface $f$, being

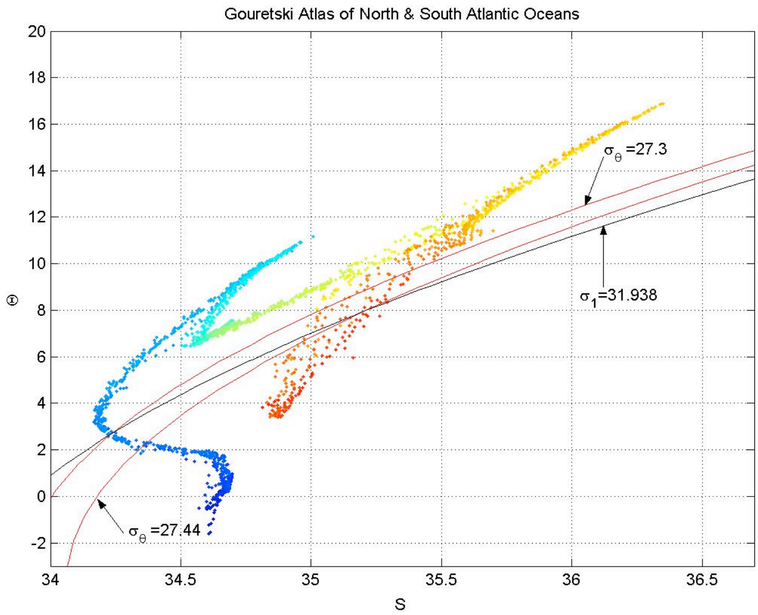
$$f\left(S_A, \Theta\right) = 0 , \qquad \text{(Twiggy\_03)}$$

which is the equation for a single line on the $(S_A, \Theta)$ diagram; a single "water-mass" for the whole world ocean. In this case, changes in $S_A$ are locally proportional to those of $\Theta$ so that $\nabla S_A \times \nabla\Theta = \mathbf{0}$ which also guarantees our required relation $\nabla P \cdot \nabla S_A \times \nabla\Theta = 0$.

Hence we have proven that the skinniness of the ocean hydrography in $(S_A, \Theta, p)$ space is a direct indication of the smallness of neutral helicity $H^n = \beta^\Theta T_b^\Theta \nabla P \cdot \nabla S_A \times \nabla\Theta$.

## The skinny nature of the ocean; demonstrated from data at constant pressure

The diagram below is a cut at constant pressure through the above three-dimensional $S_{\mathrm{A}} - \Theta - p$ data. The cut is at a pressure of $500\ \mathrm{dbar}$. This diagram illustrates the smallness of neutral helicity from the perspective of the equation $H^{\mathrm{n}} = P_z \beta^{\Theta} T_{\mathrm{b}}^{\Theta} \left( \nabla_n S_{\mathrm{A}} \times \nabla_n \Theta \right) \cdot \mathbf{k}$ .



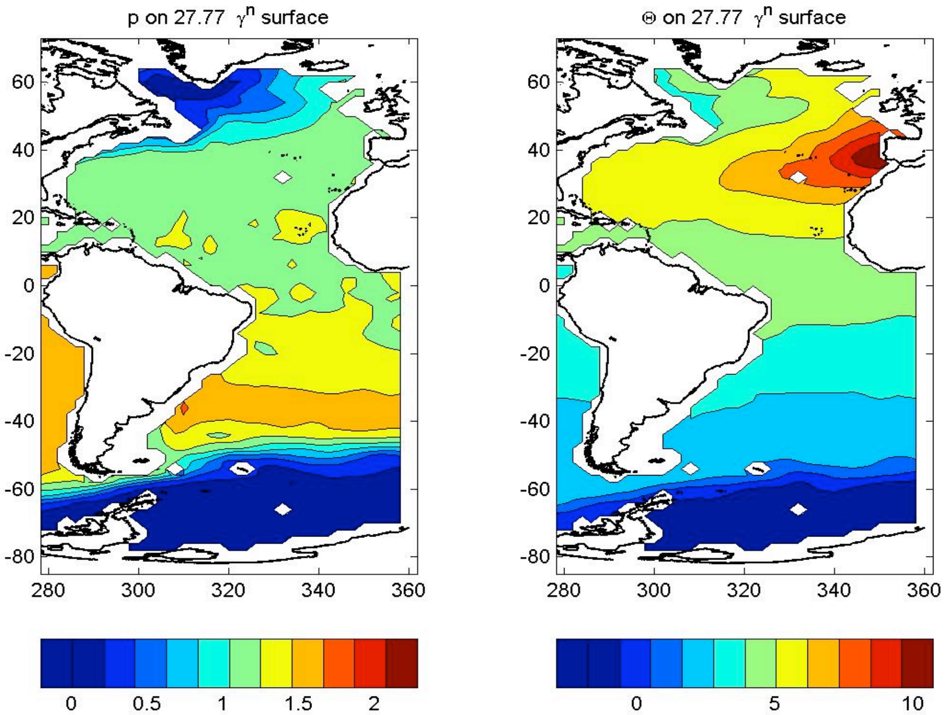
# The skinny nature of the ocean; demonstrated from data on Neutral Density surfaces

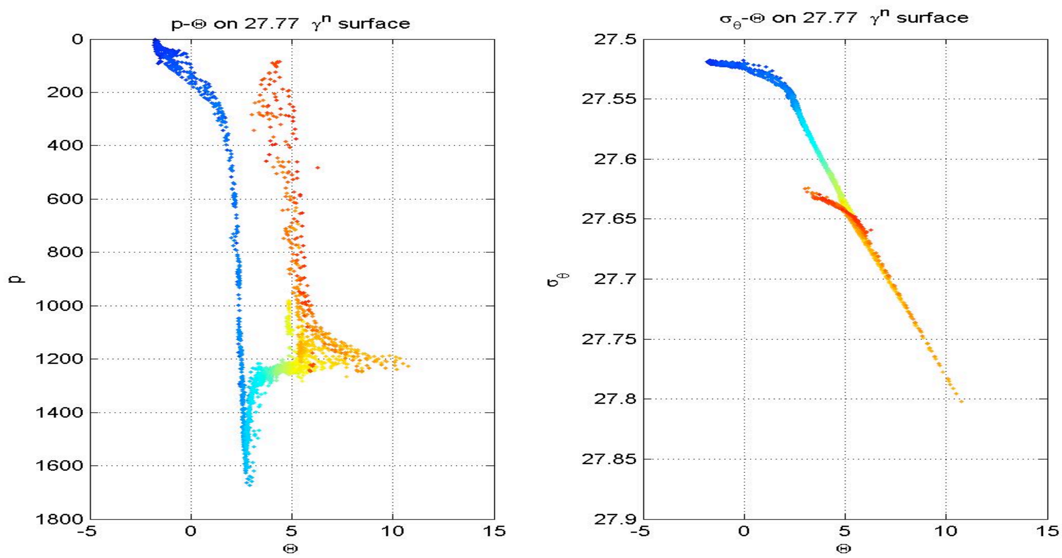
Here the "skinny" nature of the ocean will be demonstrated by looking at data on approximately neutral surfaces; Neutral Density $\gamma^{\mathrm{n}}$ surfaces. The following lines of the equation for neutral helicity

$$\begin{aligned} H^{\mathrm{n}} &= g^{-1} N^2 T_{\mathrm{b}}^{\Theta} \left( \nabla_n P \times \nabla_n \Theta \right) \cdot \mathbf{k} \\ &\approx g^{-1} N^2 T_{\mathrm{b}}^{\Theta} \left( \nabla_a P \times \nabla_a \Theta \right) \cdot \mathbf{k} \end{aligned} \qquad (3.13.2)$$

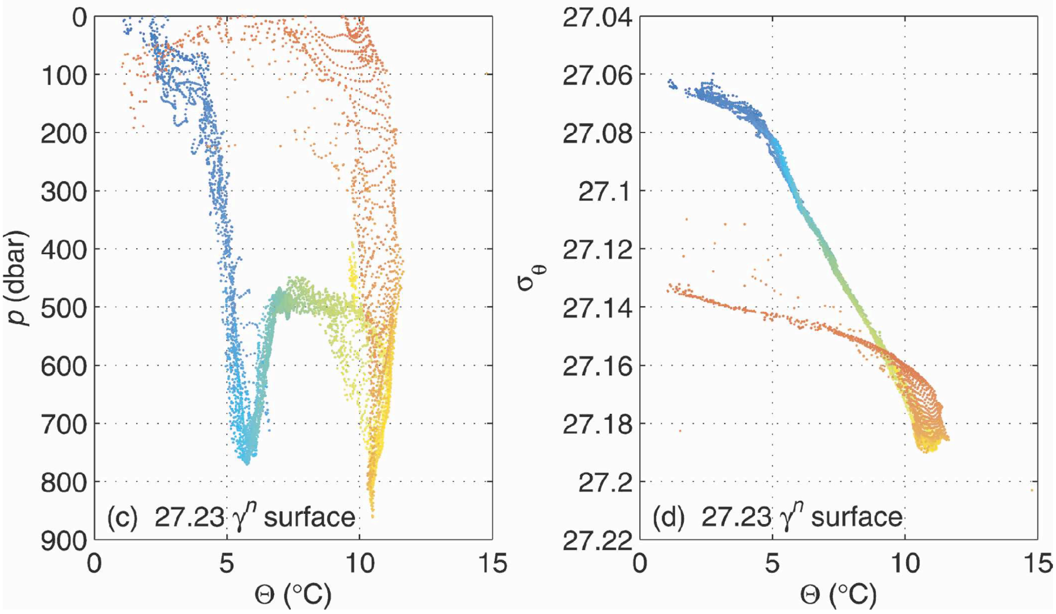
show that neutral helicity $H^{\mathrm{n}}$ will be small if the contours of $P$ and of $\Theta$ on a $\gamma^{\mathrm{n}}$ surface are lined up; that is if $\nabla_a P$ and $\nabla_a \Theta$ are parallel.

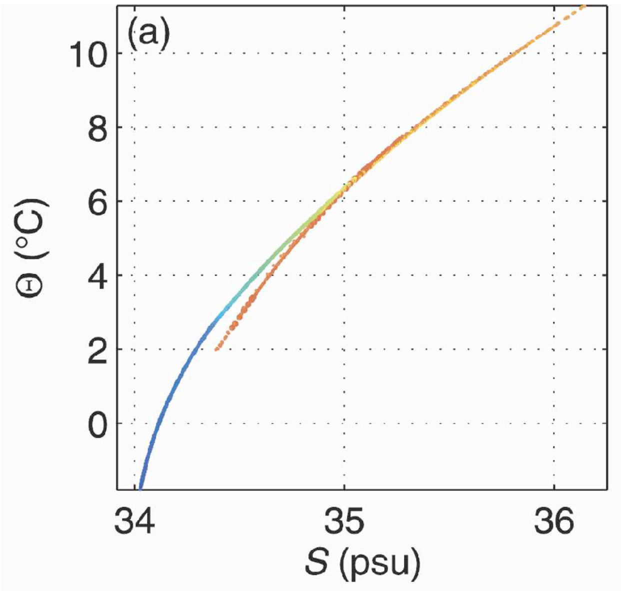
The ocean seems desperate to minimize $H^{\mathrm{n}}$; either $\nabla_a P$ and $\nabla_a \Theta$ are parallel or where they are not parallel, one of $\nabla_a P$ or $\nabla_a \Theta$ is tiny.




Notice the rather large range of potential density of $0.28\,\mathrm{kg\,m^{-3}}$ on this Neutral Density surface. Also, the value of potential density at the northern hemisphere outcrop is larger than that at the southern hemisphere outcrop by about $0.1\,\mathrm{kg\,m^{-3}}$.




The above plots confirm that the ocean is rather "skinny" in $\left(S_{\mathrm{A}},\Theta,p\right)$ space and hence that neutral helicity $H^{\mathrm{n}}$ is small in some sense (small compared to what?).

Note that while for some purposes a zero-neutral-helicity ocean,

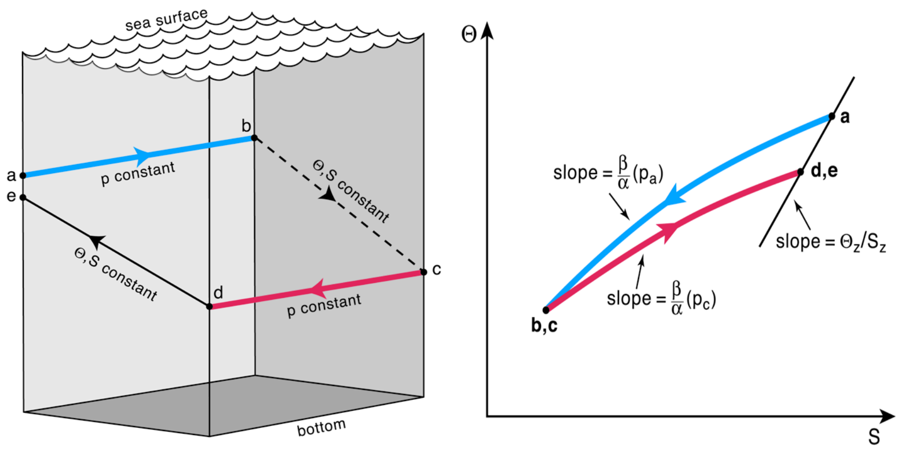
$$f\left(S_{\mathrm{A}},\Theta,p\right)=0 \qquad \text{(Twiggy\_01)}$$

might be a reasonable approximation, this $f\left(S_{\mathrm{A}},\Theta,p\right)=0$ surface is multi-valued along any particular axis. We saw this on the rotating view of the data in three $\left(S_{\mathrm{A}},\Theta,p\right)$ dimensions. This multi-valued nature is also apparent on the last figure which is of only one approximately neutral surface. A slightly denser surface would have the same $\left(S_{\mathrm{A}},\Theta\right)$ values in the Southern Atlantic as the above plot has in the North Atlantic.

Note also in the above figures that where a particular Neutral Density surface comes to the surface (outcrops) in the North Atlantic, it has a greater potential density than in the Southern Ocean by between $0.07\ \mathrm{kg\,m^{-3}}$ and $0.14\ \mathrm{kg\,m^{-3}}$. This is a general feature of the ocean; approximately neutral surfaces have different potential densities even at the reference pressure of that potential density. The northern hemisphere and southern hemisphere parts of a single ocean are separate branches in these multi-valued spaces.

## Consequences of non-zero neutral helicity

This diagram below is a simple example of the ill-defined nature of a "neutral surface" and the implication for mean dianeutral motion. The lateral mixing which causes the changes of $S_A$ and $\Theta$ along this path occur at very different pressures. It is the rotation of the isopycnals on the $S_A - \Theta$ diagram (because of the different pressures) that causes the ill-defined nature of "neutral surfaces", that is, the helical nature of neutral trajectories. In this example $\nabla_a P$ and $\nabla_a \Theta$ are at right angles, that is, $\nabla_a P \cdot \nabla_a \Theta = 0$.




The cork-screwing motion as fluid flows along a helical neutral trajectory causes vertical dia-surface flow through any well-defined density surface. This mean diapycnal flow occurs in the absence of any vertical mixing process. That is, this mean vertical advection occurs in the absence of the dissipation of turbulent kinetic energy, and is additional to the other dianeutral advection processes, thermobaricity and cabbeling.